Laminar Taylor-Couette Flow

14/11/2008

Giacomo Lorenzoni

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Laminar_Taylor-Couette_flow/

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/PEEIapplDown.aspx?var=12

Laminar Taylor-Couette flow

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units

Coordinate system: Cylindrical

Coordinates: $\underline{c}$ of which $\underline{c}\equiv\{c_i;i=1,3\}$ $\underline{\Re}\langle c_1\rangle\equiv[0,\infty)$ $\underline{\Re}\langle c_2\rangle\equiv[0,2\cdot\pi)$ $\underline{\Re}\langle c_3\rangle\equiv(-\infty,\infty)$

Coordinate versors: $\{\boldsymbol{\kappa}_i;i=1,3\}$

Unknown functions: $\{w_1,w_2,w_3,P\}$ of which $\mathbf{w}\equiv\Sigma_{i=1,3}(w_i\cdot\boldsymbol{\kappa}_i)$, $[w_i]\equiv$[speed], $\mathbf{w}$ the velocity vector, $[P]\equiv$[pressure].

Deduction of the differential analytical model:

The $c_1=c_1(\underline{x})\equiv(x_1^2+x_2^2)^{0.5}$ $c_2=c_2(\underline{x})\equiv\arcsin(x_2/(x_1^2+x_2^2)^{0.5})=\arctan\langle x_2/x_1\rangle+K$ $c_3=c_3(\underline{x})\equiv x_3$ of which $\underline{x}\equiv\{x_i;i=1,3\}$ $\underline{\Re}\langle x_i\rangle\equiv(-\infty,\infty)$, imply the $x_1=x_1(\underline{c})\equiv c_1\cdot\cos\langle c_2\rangle$ $x_2=x_2(\underline{c})\equiv c_1\cdot\sin\langle c_2\rangle$ $x_3=x_3(\underline{c})\equiv c_3$, where $\underline{x}$ are the coordinates of a Cartesian coordinate system that have the versors $\{\boldsymbol{\upsilon}_i;i=1,3\}$, and K is a constant determined by the trigonometric quadrant of $\{x_1,x_2\}$. Hence

$\partial c_1/\partial x_1=\cos\langle c_2\rangle=x_1/c_1$ $\partial c_1/\partial x_2=\sin\langle c_2\rangle=x_2/c_1$ $\partial c_2/\partial x_1=-\sin\langle c_2\rangle/c_1=-x_2/c_1^2$
$\partial c_2/\partial x_2=\cos\langle c_2\rangle/c_1=x_1/c_1^2$ $\partial c_1/\partial x_3=\partial c_2/\partial x_3=\partial c_3/\partial x_1=\partial c_3/\partial x_2=0$ $\partial c_3/\partial x_3=1$

$$\partial_{hi}\equiv\partial c_h/\partial x_i=\hat{o}_{h1i3}\cdot\hat{o}_{h2i3}\cdot\hat{o}_{h3i1}\cdot\hat{o}_{h3i2}\cdot((\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)/c_1+(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)/c_1^2+\breve{o}_{h3i3})$$

$$\partial^2_{hij}\equiv\partial^2 c_h/\partial x_i\partial x_j=\hat{o}_{h1i3}\cdot\hat{o}_{h1j3}\cdot\hat{o}_{h2i3}\cdot\hat{o}_{h2j3}\cdot\hat{o}_{h3i1}\cdot\hat{o}_{h3j1}\cdot\hat{o}_{h3i2}\cdot\hat{o}_{h3j2}\cdot$$
$$(\delta_{j1}\cdot(\breve{o}_{h1i1}/c_1-(\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)\cdot x_1/c_1^3+\breve{o}_{h2i2}/c_1^2-2\cdot(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)\cdot x_1/c_1^4)+$$
$$\delta_{j2}\cdot(\breve{o}_{h1i2}/c_1-(\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)\cdot x_2/c_1^3-\breve{o}_{h2i1}/c_1^2-2\cdot(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)\cdot x_2/c_1^4))$$

where $\breve{o}_{ijhk}=1-\hat{o}_{ijhk}$, $\hat{o}_{ijhk}=0$ if $i=j$ $h=k$ and otherwise $\hat{o}_{ijhk}=1$.

The $\boldsymbol{\kappa}_i\equiv\Sigma_{i=1,3}(\kappa_{ii}\cdot\boldsymbol{\upsilon}_i)$ implies $\boldsymbol{\kappa}_i\cdot\boldsymbol{\upsilon}_j=\Sigma_{i=1,3}(\kappa_{ii}\cdot\boldsymbol{\upsilon}_i\cdot\boldsymbol{\upsilon}_j)$. This and $\boldsymbol{\upsilon}_i\cdot\boldsymbol{\upsilon}_j=\delta_{ij}$ (where $\{\delta_{ij}=0;\forall i\neq j\}$ $\{\delta_{ij}=1;\forall i=j\}$), imply $\boldsymbol{\kappa}_i\cdot\boldsymbol{\upsilon}_j=\kappa_{ij}$ and then $\kappa_{ij}=\cos\langle\alpha_{ij}\rangle$ where α_{ij} is the angle between $\boldsymbol{\kappa}_i$ and $\boldsymbol{\upsilon}_j$. Hence

$\kappa_{11}=\kappa_{22}=x_1/c_1$ $\kappa_{33}=1$ $\kappa_{12}=-\kappa_{21}=x_2/c_1$ $\kappa_{13}=\kappa_{23}=\kappa_{31}=\kappa_{32}=0$

$$\kappa_{ij}=\hat{o}_{i1j3}\cdot\hat{o}_{i2j3}\cdot\hat{o}_{i3j1}\cdot\hat{o}_{i3j2}\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1/c_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2/c_1+\breve{o}_{i3j3})$$

$$\partial_{ijh}\equiv\partial\kappa_{ij}/\partial x_h=\hat{o}_{i1j3}\cdot\hat{o}_{i2j3}\cdot\hat{o}_{i3j1}\cdot\hat{o}_{i3j2}\cdot(\delta_{h1}\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})/c_1-((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_1/c_1^3)+$$
$$\delta_{h2}\cdot((\breve{o}_{i1j2}-\breve{o}_{i2j1})/c_1-((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_2/c_1^3))$$

$\partial^2_{ijhk}\equiv\partial^2\kappa_{ij}/\partial x_h\partial x_k=-\hat{o}_{i1j3}\cdot\hat{o}_{i2j3}\cdot\hat{o}_{i3j1}\cdot\hat{o}_{i3j2}\cdot(\delta_{k1}\cdot(\delta_{h1}\cdot(2\cdot(\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1/c_1^{3}+$
$((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot(c_1^{-3}-3\cdot x_1^{2}/c_1^{5}))+$
$\delta_{h2}\cdot((\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_1/c_1^{3}+(\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_2/c_1^{3}-3\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_1\cdot x_2/c_1^{5}))+$
$\delta_{k2}\cdot(\delta_{h1}\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_2/c_1^{3}+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_1/c_1^{3}-3\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_1\cdot x_2/c_1^{5})+$
$\delta_{h2}\cdot(2\cdot(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2/c_1^{3}+((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot(c_1^{-3}-3\cdot x_2^{2}/c_1^{5}))))$

The $\boldsymbol{\upsilon}_i\cdot\boldsymbol{\upsilon}_j=\delta_{ij}$ $\boldsymbol{\kappa}_i\cdot\boldsymbol{\upsilon}_j=\kappa_{ij}$ $\Sigma_{j=1,3}(v_j\cdot\boldsymbol{\kappa}_j)=\Sigma_{j=1,3}(\mathit{v}_j\cdot\boldsymbol{\upsilon}_j)$, imply

$$\mathit{v}_i=\Sigma_{j=1,3}(v_j\cdot\kappa_{ji}) \quad (1)$$

From $F(\underline{x})\equiv\mathrm{F}(\underline{c}(\underline{x}))$ where $\underline{c}(\underline{x})\equiv\{c_i(\underline{x});i=1,3\}$, follows

$$\partial F(\underline{x})/\partial x_i=\partial\mathrm{F}(\underline{c}(\underline{x}))/\partial x_i=\Sigma_{i=1,3}((\partial\mathrm{F}(\underline{c}(\underline{x}))/\partial c_i)\cdot(\partial c_i(\underline{x})/\partial x_i))\equiv\Sigma_{i=1,3}((\partial\mathrm{F}/\partial c_i)\cdot\partial_{ii}) \quad (2)$$

$$\partial^2F(\underline{x})/\partial x_i\partial x_j=\partial^2\mathrm{F}(\underline{c}(\underline{x}))/\partial x_i\partial x_j=\partial(\partial\mathrm{F}(\underline{c}(\underline{x}))/\partial x_i)/\partial x_j=\Sigma_{i=1,3}(\partial((\partial\mathrm{F}(\underline{c}(\underline{x}))/\partial c_i)\cdot(\partial c_i(\underline{x})/\partial x_i))/\partial x_j)=$$
$$\Sigma_{i=1,3}((\partial\mathrm{F}(\underline{c}(\underline{x}))/\partial c_i)\cdot(\partial^2c_i(\underline{x})/\partial x_i\partial x_j)+(\partial c_i(\underline{x})/\partial x_i)\cdot\Sigma_{j=1,3}((\partial^2\mathrm{F}(\underline{c}(\underline{x}))/\partial c_i\partial c_j)\cdot(\partial c_j(\underline{x})/\partial x_j)))\equiv$$
$$\Sigma_{i=1,3}((\partial\mathrm{F}/\partial c_i)\cdot\partial^2_{iij}+\Sigma_{j=1,3}(\partial_{ii}\cdot\partial_{jj}\cdot(\partial^2\mathrm{F}/\partial c_i\partial c_j))) \quad (3)$$

The continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity, in Cartesian coordinates are respectively

$$\Sigma_{i=1,3}(\partial\mathit{w}_i/\partial x_i)=0 \quad (4)$$

$$\{\rho\cdot(\Sigma_{j=1,3}(\mathit{w}_j\cdot(\partial\mathit{w}_i/\partial x_j))-\mathit{F}_i)+\partial P/\partial x_i-\mu\cdot\Sigma_{j=1,3}(\Sigma_{h=1,3}(\Sigma_{k=1,3}(\delta_{jikh}\cdot(\partial^2\mathit{w}_k/\partial x_h\partial x_j))))=0;i=1,3\} \quad (5)$$

of which: $\mathbf{w}=\Sigma_{i=1,3}(\mathit{w}_i\cdot\boldsymbol{\upsilon}_i)$, $[\mathit{w}_i]\equiv$[speed], $[\rho]\equiv$[density], $\mathbf{F}\equiv\Sigma_{i=1,3}(\mathit{F}_i\cdot\boldsymbol{\upsilon}_i)$, $[\mathit{F}_i]\equiv$[force/mass], $\mathbf{F}$ the body force vector per unit mass, $[\mu]\equiv$[dynamic viscosity], $\delta_{ijkh}\equiv\delta_{ik}\cdot\delta_{jh}+\delta_{jk}\cdot\delta_{ih}-(2/3)\cdot\delta_{hk}\cdot\delta_{ij}$.

From $\Sigma_{i=1,3}(\mathit{w}_i\cdot\boldsymbol{\upsilon}_i)=\Sigma_{i=1,3}(w_i\cdot\boldsymbol{\kappa}_i)$ and (1) follows $\mathit{w}_i=\Sigma_{j=1,3}(w_j\cdot\kappa_{ji})$. This, (4) (5) (2) and (3), imply (6) and (7).

Differential analytical model:

$$\Sigma_{i=1,3}(\Sigma_{j=1,3}(\partial_{jii}\cdot w_j+\Sigma_{h=1,3}(\kappa_{ji}\cdot\partial_{hi}\cdot(\partial w_j/\partial c_h))))=0 \quad (6)$$

$$\{\Sigma_{j=1,3}(\partial_{ji}\cdot(\partial P/\partial c_j)+$$
$$\Sigma_{h=1,3}(\Sigma_{k=1,3}(\rho\cdot\kappa_{hj}\cdot\partial_{kij}\cdot w_h\cdot w_k+$$
$$\Sigma_{m=1,3}(\rho\cdot\kappa_{hj}\cdot\kappa_{ki}\cdot\partial_{mj}\cdot w_h\cdot(\partial w_k/\partial c_m)-\mu\cdot\delta_{jikh}\cdot\partial^2_{mkhj}\cdot w_m-$$
$$\mu\cdot\delta_{jikh}\cdot\Sigma_{n=1,3}((\partial_{nj}\cdot\partial_{mkh}+\partial_{nh}\cdot\partial_{mkj}+\kappa_{mk}\cdot\partial^2_{nhj})\cdot(\partial w_m/\partial c_n)+$$
$$\Sigma_{p=1,3}(\kappa_{mk}\cdot\partial_{nh}\cdot\partial_{pj}\cdot(\partial^2w_m/\partial c_n\partial c_p)))))))-\rho\cdot\mathit{F}_i=0;i=1,3\} \quad (7)$$

of which $\rho=998.2071$ $\mu=0.001003$. The (6) and (7) are respectively, in cylindrical coordinates, the continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity.

Known functions: $\{F_i;i=1,156\}$ of which $\{F_i\equiv\mathit{F}_i=0;i=1,3\}$ $F_{A\langle h,i\rangle}=\partial_{hi}$ $F_{B\langle h,i,j\rangle}=\partial^2_{hij}$ $F_{C\langle i,j\rangle}=\kappa_{ij}$ $F_{D\langle i,j,h\rangle}=\partial_{ijh}$ $F_{E\langle i,j,h,k\rangle}=\partial^2_{ijhk}$ $A_{hi}=3+h+3\cdot(i-1)$ $B_{hij}=A_{33}+h+3\cdot(i-1)+9\cdot(j-1)$ $C_{ij}=B_{333}+i+3\cdot(j-1)$ $D_{ijh}=C_{33}+i+3\cdot(j-1)+9\cdot(h-1)$ $E_{ijhk}=D_{333}+i+3\cdot(j-1)+9\cdot(h-1)+27\cdot(k-1)$.

Definition set: $\{\underline{c}\,/\,R_1\leq c_1\leq R_2;0\leq c_2<2\cdot\pi;0\leq c_3\leq L_3\}$ $R_1=1$ $R_2=4$ $L_3=1000$.

Conditions: $w_1(\underline{c})=w_3(\underline{c})=\partial w_2/\partial c_2=\partial P(\underline{c})/\partial c_2=0$ $w_2(R_1,c_2,c_3)=\mathbb{W}_I=1$ $w_2(R_2,c_2,c_3)=\mathbb{W}_E=0.9$ $P(R_1,c_2,c_3)=\mathbb{P}_I=1000000$ (8)

Related files: mad.txt

Exact solution:

From here follows

$w_2=w_2(c_1)\equiv(\mathbb{W}_I\cdot(R_2/c_1-c_1/R_2)+\mathbb{W}_E\cdot(c_1/R_1-R_1/c_1))/(R_2/R_1-R_1/R_2)$ (9)

The equilibrium between pressure and centrifugal forces (and the $\partial P(\underline{c})/\partial c_2=\partial P(\underline{c})/\partial c_3=0$ and (9)) imply $dP=(w_2^2/c_1)\cdot dc_1$. From this and (9) follows

$P=\rho\cdot(2^{-1}\cdot C^2\cdot c_1^2+2\cdot B\cdot C\cdot \ln(c_1)-2^{-1}\cdot B^2\cdot c_1^{-2})\cdot A^{-2}+D$ (10)

where $A\equiv(R_2/R_1-R_1/R_2)$ $B\equiv(\mathbb{W}_I\cdot R_2-\mathbb{W}_E\cdot R_1)$ $C\equiv(\mathbb{W}_E/R_1-\mathbb{W}_I/R_2)$ $D\equiv\mathbb{P}_I-\rho\cdot(2^{-1}\cdot C^2\cdot R_1^2+2\cdot B\cdot C\cdot \ln(R_1)-2^{-1}\cdot B^2\cdot R_1^{-2})\cdot A^{-2}$

Note: In the following diagrams, the continuous line and the symbol ● (full circle) are respectively inherent to the (9) and (10), and the solution calculated by PEEI.

Case 1: points-1.txt, points-1.bin, cond-1.txt, sol-1.txt, plot-1-1.jpg, plot-1-2.jpg

Case 2: points-2.txt, points-2.bin, cond-2.txt, sol-2.txt, plot-2-1.jpg, plot-2-2.jpg

Case 3: points-3.txt, points-3.bin, cond-3.txt, sol-3.txt, plot-3-1.jpg, plot-3-2.jpg

Case 4: points-4.txt, points-4.bin, cond-4.txt, sol-4.txt, plot-4-1.jpg, plot-4-2.jpg

Case 5: points-5.txt, points-5.bin, cond-5.txt, sol-5.txt, plot-5-1.jpg, plot-5-2.jpg

Case 6: points-6.txt, points-6.bin, cond-6.txt, sol-6.txt, plot-6-1.jpg, plot-6-2.jpg

Case 7: points-7.txt, points-7.bin, cond-7.txt, sol-7.txt, plot-7-1.jpg, plot-7-2.jpg

Case 8: points-8.txt, points-8.bin, cond-8.txt, sol-8.txt, plot-8-1.jpg, plot-8-2.jpg

Case 9: points-9.txt, points-9.bin, cond-9.txt, sol-9.txt, plot-9-1.jpg, plot-9-2.jpg

Case 10: points-10.txt, points-10.bin, cond-10.txt, sol-10.txt, plot-10-1.jpg, plot-10-2.jpg

Case 11: points-11.txt, points-11.bin, cond-11.txt, sol-11.txt, plot-11-1.jpg, plot-11-2.jpg

Case 12: points-12.txt, points-12.bin, cond-12.txt, sol-12.txt, plot-12-1.jpg, plot-12-2.jpg

Case 13: points-13.txt, points-13.bin, cond-13.txt, sol-13.txt, plot-13-1.jpg, plot-13-2.jpg

Case 14: points-14.txt, points-14.bin, cond-14.txt, sol-14.txt, plot-14-1.jpg, plot-14-2.jpg

Case 15: points-15.txt, points-15.bin, cond-15.txt, sol-15.txt, plot-15-1.jpg, plot-15-2.jpg

Case 16: points-16.txt, points-16.bin, cond-16.txt, sol-16.txt, plot-16-1.jpg, plot-16-2.jpg

Case 17: points-17.txt, points-17.bin, cond-17.txt, sol-17.txt, plot-17-1.jpg, plot-17-2.jpg

Case 18: points-18.txt, points-18.bin, cond-18.txt, sol-18.txt, plot-18-1.jpg, plot-18-2.jpg

Case 19: points-19.txt, points-19.bin, cond-19.txt, sol-19.txt, plot-19-1.jpg, plot-19-2.jpg

Case 20: points-20.txt, points-20.bin, cond-20.txt, sol-20.txt, plot-20-1.jpg, plot-20-2.jpg

Case 21: points-25.txt, points-25.bin, cond-25.txt, sol-25.txt, plot-25-1.jpg, plot-25-2.jpg

Case 22: points-30.txt, points-30.bin, cond-30.txt, sol-30.txt, plot-30-1.jpg, plot-30-2.jpg

Case 23: points-35.txt, points-35.bin, cond-35.txt, sol-35.txt, plot-35-1.jpg, plot-35-2.jpg

Case 24: points-40.txt, points-40.bin, cond-40.txt, sol-40.txt, plot-40-1.jpg, plot-40-2.jpg

Case 25: points-45.txt, points-45.bin, cond-45.txt, sol-45.txt, plot-45-1.jpg, plot-45-2.jpg

Case 26: points-50.txt, points-50.bin, cond-50.txt, sol-50.txt, plot-50-1.jpg, plot-50-2.jpg

Case 27: points-60.txt, points-60.bin, cond-60.txt, sol-60.txt, plot-60-1.jpg, plot-60-2.jpg